J. Appl. Maths Mechs, Vol. 55, No. 1, pp. 42-48, 1991
Printed in Great Britain

0021-8928/91 \$15.00+0.00

SINGULAR TRAJECTORY IN THE PROBLEM OF SIMPLE PURSUIT ON A MANIFOLD*

A.A. MELIKYAN and N.V. OVAKIMYAN

Games with at most two minimal geodesics connecting two points in their phase space (manifold) are considered. It is shown that singular trajectories - envelopes of geodesics - may develop in such games. A necessary condition for the existence of singular motions (non-emptiness of the set (3.5)) is derived as a certain requirement from the geometrical properties of the phase manifold of the game. An algorithm to construct the singular motions is proposed and the Hamiltonian equations describing these motions are given. The sufficiency of the proposed construction is investigated numerically for particular examples. The paper generalizes and extends the previous study /1/.

In a number of pursuit games, the optimal trajectories of the players are geodesics in the space of phase states of the game /2, 3/. This is so, in particular, when the players are represented by velocity-controlled points in Euclidean space with spherical constraints on the magnitude of the velocities. In this case, there always exists a unique geodesic (a straight segment) connecting any two locations of the players; the optimal velocities of the players are directed along this connecting line. The geodesic property of optimal trajectories may be lost if the connecting geodesic is non-unique for some set of player positions. This case is observed, for instance, in the presence of a bounded obstacle, when the players are barred from reaching a bounded region in the Euclidean phase space /1/.

1. *Simple pursuit game*. Assume that the points (players) P and E in some n-dimensional manifold M (the space of the game) move according to the equations

$$\begin{aligned} &P\colon x^{\cdot} = u,\ u \in E_1(x);\\ &E\colon y^{\cdot} = v,\ v \in E_\nu(y),\ 0 < \nu < 1\\ &E_\alpha(x) = \{u \in R^n\colon (G(x)u, u) \leqslant \alpha^2\} \end{aligned} \tag{1.1}$$

where $x, y \in R^n$ are the local coordinates of the points P and E, $E_\alpha(x)$, $\alpha \geqslant 0$, is an ellipsoid. The positive definite matrix $G(x)$ is a metric tensor of the manifold M; the scalar product is denoted by parentheses.

Thus, the magnitude of the velocity of the point P does not exceed unity and the velocity of the point E does not exceed $\nu > 0$. Since $\nu < 1$, the point P in general has an opportunity to reduce the distance $L(x, y)$ between P and E. The objective of player P is to ensure the fastest possible approach of the points P and E to a distance l, i.e., to satisfy the condition

$$L(x(T), y(T)) \leqslant l \quad (l \geqslant 0) \tag{1.2}$$

for some instant of time $T > 0$; zero is taken as the initial instant. Player E attempts to maximize the capture time T. We assume that $L(x(0), y(0)) > l$. The non-negative number l is called the capture radius /2/.

The distance between points P and E is determined by means of the following variational problem /4/:

$$L(x, y) = \min_{\xi(\cdot)} \int_{s_0}^{s_1} \sqrt{(G(\xi)\xi^{\cdot}, \xi^{\cdot})}\, ds, \quad \xi(s_0) = x, \quad \xi(s_1) = y \tag{1.3}$$

where the minimum is over all piecewise-smooth curves joining the points P and E: $\xi(\cdot) = \{\xi(s)\colon s_0 \leqslant s \leqslant s_1\}$.

We will assume that the global minimum in (1.3) is achieved for any pair of points of the manifold. The next basic assumption regarding the geometry of the manifold is that the global minimum in (1.3), if it is non-unique, can be represented in the form

*Prikl.Matem.Mekhan.,55,1,54-62,1991

$$L(x, y) = \min [L^+(x, y), L^-(x, y)] \tag{1.4}$$

where $L^\pm$ are the distances at which local minima are reached in (1.3) (Fig.1); the functions $L^\pm(x, y)$ are smooth and are defined in some neighbourhood of the pair (x, y), which differs for each pair. Assumption (1.4) is satisfied, for instance, in the case when the points move in a Euclidean plane that contains a bounded convex obstacle /1/ or on two-dimensional surfaces of revolution in a three-dimensional space.

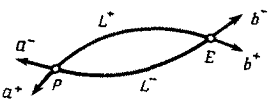

Fig.1

Relationships (1.1) and (1.2) describe a differential game with payoff function (functional) equal to the time of the process. The game may be treated as a combination of two optimal games with guaranteed results for players P and E /5/; the minimax (maximin) problem uses positional controls of player P (E) and piecewise-continuous functions of time for player E (P). The optimal positional controls of the players are optimal (ε-optimal) guaranteeing strategies in one of these games. The procedure for constructing the trajectory (the motion) for a given pair of controls is described in /5/.

For some subset of positions (x, y), the motion with velocities of maximum magnitude along the tangent to the geodesic is optimal for both players. This is established using the following relationships. Consider the time derivative of the distance (1.3) in the direction of motion of system (1.1),

$$L^{\cdot} = (L_x, u) + (L_y, v);\ L_x = G(x)\, a,\ L_y = G(y)\, b \tag{1.5}$$
$$a = a(x, y) = -\xi^{\cdot}(s_0)/\,|\,\xi^{\cdot}(s_0)\,|_x$$
$$b = b(x, y) = \xi^{\cdot}(s_1)/\,|\,\xi^{\cdot}(s_1)\,|_y;\ \ |\,\eta\,|_x \equiv \sqrt{(G(x)\,\eta,\ \eta)}$$

Here a and b are the unit vectors tangent to the geodesic at its two ends and pointing outward (Fig.1). The formulas for the partial derivatives L_x and L_y in (1.5) follow from the form of the first variation of the functional (1.3) /4/. Evaluation of the minimax or the maximin of the derivative (1.5) subject to the constraints (1.1) gives

$$\min_u \max_v L^{\cdot} = \min_u (L_x,\ u) + \max_v (L_y,\ v) = -1 + \nu \tag{1.6}$$
$$u^*(x, y) = -a(x, y),\ v^*(x, y) = \nu\, b(x, y)$$

Here u^* and v^* are the unique vectors on which the extrema in (1.6) are reached.

Thus, the function

$$S(x, y) = [L(x, y) - l]/(1 - \nu) \tag{1.7}$$

equal to the time of pursuit along the geodesic satisfies the basic equation of the theory of differential games at the points of differentiability of the length $L(x, y)$ /5/:

$$\min_u \max_v S^{\cdot} = \max_v \min_u S^{\cdot} = -1 \tag{1.8}$$
$$u \in E_1(x),\ \ v \in E_\nu(y)$$

Using relationships (1.6)-(1.8), we can show the strategy (1.6) guarantees for player P a pursuit time not exceeding (1.7); if representation (1.4) exists, the strategy may be based on either function L^+, L^-. For player E, however, strategy (1.6) in general does not guarantee the time (1.7). This is so, for instance, when M is the n-dimensional Euclidean space R^n and the formulas take the simple form

$$L(x, y) = |\,x - y\,|;\ u^*(x, y) = (y - x)/\,|\,y - x\,|$$
$$v^*(x, y) = \nu\,(y - x)/\,|\,y - x\,|$$

where $|\,x\,| = \sqrt{(x,\ x)}$ is the vector Euclidean norm. This is asymmetry between the players is attributed to the fact that player P minimizes a quantity proportional to L while (1.4) also requires minimization.

Property (1.4) in general allows player P to guarantee, for some subset of positions (x, y), a pursuit time less than (1.7); the pursuit strategy on this subset may differ from the control that recommends along the geodesic.

2. *The necessary condition for optimality.* Denote the product $M \times M$ by Z; the points of the manifold Z will be denoted by z, $z = (x, y) \in Z$. Condition (1.2) defines in Z a so-called terminal set $Z_T \subset Z$. It can be shown that in a sufficiently small neighbourhood of the set Z_T, the optimal result(the value of the game) is $S(z)$ (1.7) guaranteed by the strategies (1.6).

We assume that a continuous value $V(z), z \in Z \setminus Z_T$, exists in the game (1.1), (1.2). The unknown region where $S(z)$ is the value of the game will be denoted by $Z_1 \subset Z$, $S(z) = V(z)$, $z \in Z_1$. The function $S(z)$ and the strategies (1.6) for $z \in Z_1$ are called a primary solution. Denote by Γ_0 the subset of Z where the minimum (1.3) is attained on two extremals, i.e., we have the equality $L^+(z) = L^-(z)$ for the functions from (1.4). The intersection $\Gamma_0 \cap Z_1$ is, in general, non-empty. The necessary condition (1.8) cannot be used to identify the points of the set Γ_0 that are contained in Z_1, because the function $S(z)$ in general is non-differentiable for $z \in \Gamma_0$ even for smooth $L^\pm(z)$. Retaining the notation $V^\cdot$ for the derivative of the value $V(z)$ with respect to the direction $(u, v) \in R^{2n}$, $V^\cdot = V^\cdot(z, u, v)$, we can write the following generalized necessary conditions of optimality /5/:

$$\min_u \max_v V^\cdot(z, u, v) \geqslant -1 \geqslant \max_v \min_u V(z, u, v) \qquad (2.1)$$
$$u \in E_1(x), \ v \in E_\nu(y)$$

For some classes of games, conditions of the form (2.1) together with boundary conditions are necessary and sufficient conditions for the optimality of the function $V(z)$ /5/. In this paper, relationships (2.1) are used as local necessary conditions of optimality.

Without giving a complete proof, we will elucidate the meaning of relationship (2.1). Reverse strict inqualities in (2.1) are ruled out: if the left-hand reverse inequality were violated, player P, say, could apply for a sufficiently short time in the neighbourhood of the point z of a fixed control $u(z)$ on which the outer minimum would be attained and the player would thus achieve a strictly better result in the game, contradicting the assumption of the optimality of $V(z)$.

At the points of smoothness of the function $V(z)$, relationships (2.1) reduce to equalities (1.8) for the function V. Evaluation of the extrema in (1.8) using the notation $p = (V_x, V_y) \in R^{2n}$ leads to the basic Bellman-Isaacs equation of the form

$$F(z, p) + 1 = 0; \ F(z, p) \equiv -\sqrt{(G^{-1}(x) V_x, V_x)} + \nu\sqrt{(G^{-1}(y) V_y, V_y)} \equiv -|\eta|_x + \nu|\psi|_y; \qquad (2.2)$$
$$\eta = \eta(z, p) = G^{-1}(x) V_x, \ \psi = \psi(z, p) = G^{-1}(y) V_y;$$
$$u^*(z, p) = -\eta/|\eta|_x, \ v^*(z, p) = \nu\psi/|\psi|_y$$

From equalities (1.5) we obtain the following Hamilton-Jacobi equations (eikonal equations) for the geodesic distance (1.3) /4/:

$$(G^{-1}(x) L_x, L_x) = 1, \ (G^{-1}(y) L_y, L_y) = 1 \qquad (2.3)$$

By equalities (2.3), the function (1.7) satisfies Eq.(2.2). The optimal motions in the game are determined in the region of smoothness of the function $S(z)$ by the characteristic equations corresponding to the partial differential Eq.(2.2) /4/:

$$z^\cdot = F_p, \ p^\cdot = -F_z; \ z = (x, y) \in R^{2n}, \qquad (2.4)$$
$$p = (V_x, V_y) \in R^{2n}$$

3. *The edge of manifold* Γ. *Structure of the solution.* We will use relationships (2.1) to identify the points of the set Γ_0 that are contained in Z_1. We denote by $a^\pm(z)$, $b^\pm(z)$ the unit tangent vectors to two geodesics at their ends (Fig.1, see (1.5)). The directional derivative of the function (1.7) at the points of the set Γ_0 using (1.4), (1.5), takes the form

$$S^\cdot = \min[S^{+\cdot}, S^{-\cdot}] \qquad (3.1)$$
$$S^{\pm\cdot}(z) = [(L_x^\pm(z), u) + (L_y^\pm(z), v)]/(1 - \nu)$$

The functions $S^\pm$ have the form (1.7) with $L^\pm$ substituted for L.

Lemma 1. The maximin (2.1) at the points of the manifold Γ_0 satisfies the condition

$$\max_v \min_u \min[S^{+\cdot}, S^{-\cdot}] = [-1 + \nu|b^+ + b^-|_y/2]/(1 - \nu) \leqslant -1$$
$$v \in E_\nu(y), \ u \in E_1(x)$$

The last inequality follows from the condition $|b^+ + b^-|_y < 2$, which is obvious for unit vectors b^+, b^-. Thus, all the points of the set Γ_0 satisfy the right-hand condition in (2.1).

Theorem 1. The minimax (2.1) for $z \in \Gamma_0$ can be represented in the form

$$\min_u \max_v \min\,[S^{+*},\ S^{-*}] = \min\,[-1,\ \Phi(z)], \qquad (3.2)$$
$$u \in E_1(x),\ v \in E_\nu(y);$$
$$\Phi(z) \equiv \frac{1}{2(1-\nu)}\,[-|a^+ + a^-|_x + \nu\,|b^+ + b^-|_y] \equiv F(z, R_z(z));$$
$$R(z) = {}^1/_2\,(S^+(z) + S^-(z))$$

and the minimum over u is achieved on a unique vector $u^* = -(a^+ + a^-)/\,|a^+ + a^-|_x$ for $\Phi(z) < -1$ and on two vectors $u^* = -a^+, u^* = -a^-$ for $\Phi(z) > -1$. For $\Phi(z) = -1$, all three vectors produce a minimum. At the points of the intersection $\Gamma_0 \cap Z_1$ we have the inequality

$$\Phi(z) + 1 \equiv F(z,\ R_z(z)) + 1 \geqslant 0 \qquad (3.3)$$

Thus, the primary solution region Z_1 includes only a part Γ_1 of the manifold Γ_0:

$$\Gamma_1 = \{z \in Z\colon\ S^+(z) = S^-(z),\ \Phi(z) \geqslant -1\} \qquad (3.4)$$

Denote by B the edge of the manifold Γ_1, whose points satisfy the equalities

$$|\,a^+(z) + a^-(z)\,|_x - \nu\,|\,b^+(z) + b^-(z)\,|_y = 2\,(1-\nu), \qquad (3.5)$$
$$L^+(z) = L^-(z)$$

Condition (3.5) enables us to define a manifold M in which the velocity-controlled pursuer can reduce the capture time (1.7) by exploiting the special geometry of the manifold, which restricts the manoeuverability of the evader.

If the phase space of the game is a circular cylinder, say, we can show that the set (3.5) is empty and Γ_1 is identical with Γ_0. In the pursuit game in a Euclidean plane with a bounded convex obstacle /1/ or on a convex two-dimensional cone, the set B is non-empty (see Sect.5).

Assume that the set B is non-empty. Assume, as in /1/, that the manifold B is the edge of two branches Γ^+, Γ^- of a singular equivocal surface, as shown in the diagram in Fig.2.

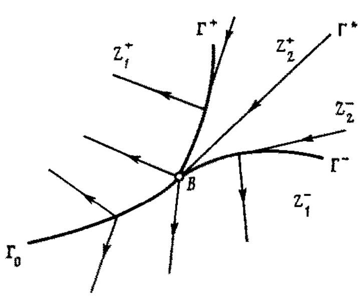

Fig.2

An equivocal surface /1, 3, 6/ is a switching (discontinuity) surface of optimal controls of the two players which contains singular optimal trajectories. If one of the players (that controlling the given surface /6/) does not switch when the phase point reaches the equivocal surface, optimal sliding along the surface is obtained; if the player switches, the optimal trajectory transversely recedes in the opposite direction from the surface. In problems with simple motion of the players, when the extrema (1.8) are reached on unique vectors, the optimal trajectories necessarily approach the equivocal surface with tangency /6/.

The space Z can be represented in the form of the sum $Z = Z_1 + \Gamma + Z_2$, $\Gamma = \Gamma^+ + \Gamma^-$; for the points $z \in Z_2$, player P strictly improves the result (1.7). This hypothesis concerning the structure of solutions of the game (1.1), (1.2) relies on the necessary conditions of optimality (1.8), (2.3), (2.2), (3.3) and the conditions of the following section: whether this system of necessary conditions is complete, i.e., produces a unique solution, must be separately proved in each particular case, in general by numerical integration of the equations of the singular characteristics.

4. ***Singular characteristic equations.*** Thus, the following necessary optimality conditions in equality form are satisfied on the unknown surface Γ:

$$F_0(z,\ p) \equiv F(z,\ p) + 1 = 0,\ F_1(z,\ V) \equiv V - S(z) = 0 \qquad (4.1)$$
$$F_{-1}(z,\ p) \equiv \{F_1 F_0\} \equiv (F_p(z,\ p),\ p - q(z)) = 0\ \ (q(z) = S_z(z))$$

The first equality is a Bellman-Isaacs equation, the second is the condition of continuity of the value of the game, and the third is the condition of tangency of the surface Γ to the optimal trajectory in the region Z_2/1, 6/. Curly braces in (4.1) are Jacobi brackets in the (z, p, V) space (Poisson brackets if there is no dependence on V).

By the method developed in /7, 8/, three equalities of the form $F_i(z, p, V) = 0$ are sufficient to derive the equations of the singular trajectories forming the manifold Γ of codimension unity in Z-space. The required equations are Hamiltonian equations with the Hamiltonian (apart from a non-zero multiplier)

$$H = \{F_{-1} F_1\}\,F_0 + \{F_1 F_0\}\,F_{-1} + \{F_0 F_{-1}\}\,F_1 \qquad (4.2)$$

To construct the surface Γ, we trace from every point of the edge B the solution of the system

$$z^{\cdot} = H_p, \; p^{\cdot} = -H_z - H_V p, \; V^{\cdot} = (p, H_p)$$

having first determined for each point $z \in B$ the corresponding value $p(z)$. The branch $\Gamma^+ (\Gamma^-)$ is constructed by replacing S in (4.1) with the function $S^+ (S^-)$.

Using the functions (4.1) in (4.2) and normalizing the Hamiltonian H so that the multiplier of F_0 equals 1, we obtain the equations of singular motion in the form

$$z^{\cdot} = F_{0p}, \quad p^{\cdot} = -F_{0z} - \frac{\{F_0\{F_1F_0\}\}}{\{F_1\{F_0F_1\}\}}(p - q(z)) \tag{4.3}$$

The last equation $V^{\cdot} = (p, H_p)$ splits off. We stress that Eqs.(4.3) are written using only two functions F_0 and S, because $F_1 \equiv V - S$.

The initial values of z, $p(z)$ for system (4.3) are given by the following proposition.

Lemma 2. For $z \in B$ on the branch Γ^+ the vectors $p = q^+$, $p = (q^+ + q^-)/2 = R_z$ satisfy the equalities $F_0 = 0$, $F_{-1} = 0$ of system (4.1). On the branch Γ^- the corresponding soltuions are $p = q^-$, $p = (q^+ + q^-)/2 = R_z$.

The lemma is proved by direct computation using relationships (2.2), (2.3), (3.2), (3.5) and the symmetry of the matrix G^{-1}.

Thus,

$$p = \frac{1}{2}(q^+ + q^-) \equiv R_z \tag{4.4}$$

is the initial value of p (the conjugate vector) for constructing both branches Γ^+ and Γ^-. The fact that a common solution (4.4) exists for both branches implies that the gradient of the value $V(z)$ is continuously continuable from the region Z_2 to the edge B. Other solutions correspond to the primary solution of the problem.

If G is the identity matrix in some region, then the function F in (2.2) for this region has the form

$$F(p) = -\sqrt{p_1^2 + \ldots + p_n^2} + \nu\sqrt{p_{n+1}^2 + \ldots + p_{2n}^2} \tag{4.5}$$

i.e., it is homogeneous of degree unity as a function of the vector p. The corresponding system is simplified and takes the form /1, 6/

$$z^{\cdot} = F_p, \; p^{\cdot} = [(S_{zz}F_p, \; F_p)/(F_{pp}q, \; q)] \; (p - q) \tag{4.6}$$

where S_{zz} and F_{pp} are symmetric matrices of second partial derivatives (Hessians).

Optimal motions in the region Z_2 can be constructed by integrating system (2.4) in reverse time with the initial conditions $z = z^0$, $p = p(z^0)$, $z^0 \in \Gamma$. Another construction technique is by taking discontinuously $p^{\cdot} = -F_{0z}$ at some instant of time in the process of integrating system (4.3), i.e., passing to the system (2.4). If this jump is made at the initial instant of time, we obtain a trajectory which leaves a point on the edge B with the initial value (4.4) of the conjugate vector. The collection of these trajectories forming some surface Γ^* partitions the region Z_2 into two subregions $Z_2 = Z_2^+ + \Gamma^* + Z_2^-$. The surface Γ^* is tangent to both surfaces Γ^+ and Γ^- at the points of the edge B.

5. ***Examples.*** As the phase space of the game (1.1)-(1.4) consider a circular cylinder in the three-dimensional Euclidean space. In this case, for the vectors $a^{\pm}, b^{\pm}$ in (3.5) we have the equality $a^+ + a^- = -(b^+ + b^-)$ Since $|a^+ + a^-| < 2$, the set B defined by equality (3.5) is empty. Thus, for a cylinder, pursuit and evasion by geodesics are optimal.

Assume that the phase space of the game is the Euclidean plane with a convex bounded obstacle /1/. We can show that the set B in this case is non-empty and it is a two-dimensional manifold with an edge (equivalent to the phase space). A numerical analysis of both trajectories is carried out in /1/.

Finally, let the phase space of the game be a two-dimensional convex cone in the three-dimensional Euclidean space with the induced metric. By deforming the convex cone, we can map it to a two-sided two-sheeted plane angle (conserving the geodesic length). The analysis of the game for a plane angle is simpler, and the rest of the discussion focuses on this case. Let $l = 0$. Capture occurs when the points P and E are on the same side of the plane and coincide.

Take a Cartesian rectangular system of coordinates with the origin at the vertex of the angle and the abscissa axis pointing along the bisector (Fig.3). The angle α is between $0 < \alpha < \pi$. The geodesics are straight segments (if the points P and E are on the same sheet) or two-segment polygonal lines (if the points P and E are on different sides). Let (x_1, x_2) and (y_1, y_2) be the coordinates of the points P and E, respectively, and let the two points lie on different sides of the plane. The two shortest geodesics crossing different rays of the angle (Fig.3) have the lengths

$$L^{\pm} = \frac{1}{\sqrt{a^2+1}} [(a^2+1)(x_1^2+y_1^2+x_2^2+y_2^2) + 2(a^2-1)(x_1y_1 - x_2y_2) \mp 4a(x_2y_1 + x_1y_2)]^{1/2} \tag{5.1}$$

Here $a = \operatorname{tg} \alpha/2$. Formula (5.1) can be obtained from simple geometrical considerations. The equality $L^+ = L^-$ defining the points of the set Γ_0 (see Sect.2) gives

$$x_1/x_2 = -y_1/y_2$$

i.e., the points P and E located on different sheets lie on rays that are symmetrical about the abscissa axis. Deforming the two-sheeted angle, we can make these rays overlap with the sides of the angle or with the bisector (the abscissa axis); the latter enables us to reduce any point of the set Γ_0 to the form $P(x_1, 0), E(y_1, 0)$.

Differentiating the functions (5.1), we can find the vectors $a^{\pm}, b^{\pm}$ of the form (1.5) (G is the identity matrix) and form the first equation in (3.5). The functions (5.1) are positively homogeneous functions of degree unity of the vectors x, y, and therefore $|a^+ + a^-|$, $|b^+ + b^-|$ in (3.5) are homogeneous of degree 0. Therefore, one of the parameters x_1, y_1 defining a point of the set B, as noted above, may be set equal to a given value. Let $y_1 = 1$. Then relationship (3.5) reduces to an equation for x_1, which is solvable in finite form.

There is no characteristic size in this problem, and the normalization condition $y_1 = 1$ can be satisfied by an appropriate choice of the length scale. In the problem with an obstacle /1/, the obstacle determines a characteristic size and the problem does not have this self-similarity property.

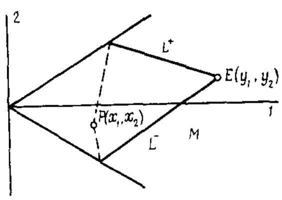

Fig.3

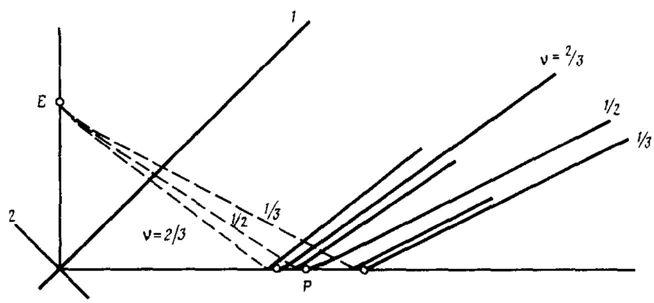

Fig.4

In general, to construct singular trajectories forming the equivocal surface, we must integrate Eq.(4.3) in reverse time with the initial conditions (see (4.4))

$$z(0) = z^0, \quad p(0) = {}^1/_2 (q^+(z^0) + q^-(z^0)), \quad z^0 \in B$$

for all points z^0 in the set B. Our analysis shows that an arbitrary point z^0 of the set B is reducible to the form $z^0 = \lambda z^*$, $z^* = (x_1^*, 0, 1, 0)$, $\lambda > 0$, where x_1^* is the root of Eq.(3.5) mentioned above. Moreover, Eqs.(4.3) with the function F (4.5) and the dependence $S(z) = L/(1-\nu)$, where L is of the form (5.1), are invariant under the change of coordinates and time $z = \lambda\xi$, $t = \lambda\tau$ with any $\lambda > 0$.

Thus, by a simple computation, the singular trajectory leaving an arbitrary point of the set B may be obtained from the standard trajectory originating from the point z^*, i.e., the complete synthesis procedure in this game reduces to the construction of two plane curves.

Such standard trajectories were constructed numerically for $a = 1, \nu = {}^1/_3, {}^1/_2, {}^2/_3$ (Fig.4). We see that the convexity properties of these trajectories are such that the collection of tangent half-lines to these trajectories uniquely sweeps the region Z_2 introduced in Sect.2. Formulas (2.2) and (1.6) can thus be used to determine the optimal positional strategies of the players in the entire space of the game. These controls are single-valued everywhere, except the scattering surface Γ_1 and the equivocal surface Γ. A complete analysis of the guaranteeing strategies in the neighbourhood of singular surfaces requires a special examination.

Calculations also show that singular trajectories of the players have straight asymptotes, to which the trajectories converge very rapidly.

REFERENCES

1. VISHNEVETSKII L.S. and MELIKYAN A.A., Optimal pursuit on a plane in the presence of an obstacle, PMM, 46, 4, 1982.
2. ISAACS R., Differential Games, Wiley, New York, 1965.
3. KRASOVSKII N.N., Game Problems on Approaching Motions, Nauka, Moscow, 1970.
4. COURANT R., Partial Differential Equations, Mir, Moscow, 1964.
5. SUBBOTIN A.I. and CHENTSOV A.G.., Optimization of Guarantee in Control Problems, Nauka, Moscow, 1981.
6. MELIKYAN A.A.., The necessary conditions of optimality on a discontinuity surface of one type in a differential game, Izv. Akad. Nauk SSSR, Tekh. Kibern., 4, 1981.
7. MELIKYAN A.A., On the construction of weak discontinuities in optimal control problems and differential games, Izv. Adad. Nauk SSSR, Tekh. Kibern. 1, 1984.
8. MELIKYAN A.A. and OVSEYEVICH A.I., Hamiltonian systems with a given invaraint manifold and some applications, PMM, 48, 2, 1984.

Translated by Z.L.

J. Appl. Maths Mechs, Vol. 55, No. 1, pp. 48-55, 1991
Printed in Great Britain

0021-8928/91 $15.00+0.00

QUASILINEAR CONFLICT-CONTROLLED PROCESSES WITH NON-FIXED TIME*

P.V. PROKOPOVICH and A.A. CHIKRII

We identify a class of conflict-controlled processes /1-3/ for which the solving functions of the group pursuit problem /4-7/ are independent of the termination time of the game while evader errors cause the process to terminate earlier than the guarantee time. Sufficient conditions are derived for the solvability of pursuit and evasion problems, and the continuity property of the solving functions is studied in detail. The sufficient conditions for the pursuit problem to be solvable do not include Pontryagin's condition /3, 8/; it is replaced with a weaker assumption related to the initial state of the process. The proposed procedure enables us to strengthen some known results on the solution of group pursuit problems.

1. The motion of a conflict-controlled object $z = (z_1, \ldots, z_n)$ in the finite-dimensional Euclidean space R^ν is described by the system of differential equations

$$z_i^{\boldsymbol{\cdot}} = A_i z_i + \varphi_i(u_i, v),\ z_i \in R^{\nu_i},\ u_i \in U_i,\ v \in V,\ z_i(0) = z_i^0 \tag{1.1}$$

Here A_i is a given square matrix of order ν_i, U_i and V are non-empty compact subsets in the spaces R^{p_i} and R^q, respectively, and the function $\varphi_i(u_i, v)$ is continuous in all its variables. Here and henceforth, $i = 1, 2, \ldots, n$.

The terminal set M^* consists of the sets M_i^*, such representable in the form

$$M_i^* = M_i^0 + M_i \tag{1.2}$$

where M_i^0 is a linear subspace of the space R^{ν_i} and M_i is a convex compact set in L_i - the orthogonal complement of M_i^0 in R^{ν_i}.

We say that the game (1.1) terminates from the initial state $z^0 = (z_1^0, \ldots, z_n^0)$ not later than in a time $T(z^0)$ if measurable functions $u_i(t) = u_i(z_i^0, v(t)) \in U_i$, $0 \leqslant t < t^*$, $t^* \leqslant T(z^0)$ exist such that $z_i(t^*) \in M_i^*$ for at least one i for any measurable function $v(t) \in V$, $0 \leqslant t \leqslant T(z^0)$, where $z_i(t)$ is the solution of the system of Eqs.(1.1) corresponding to the pair of controls $u_i(t)$ $v(t)$ and the initial state z_i^0.

*Prikl.Matem.Mekhan.,55,1,63-71,1991